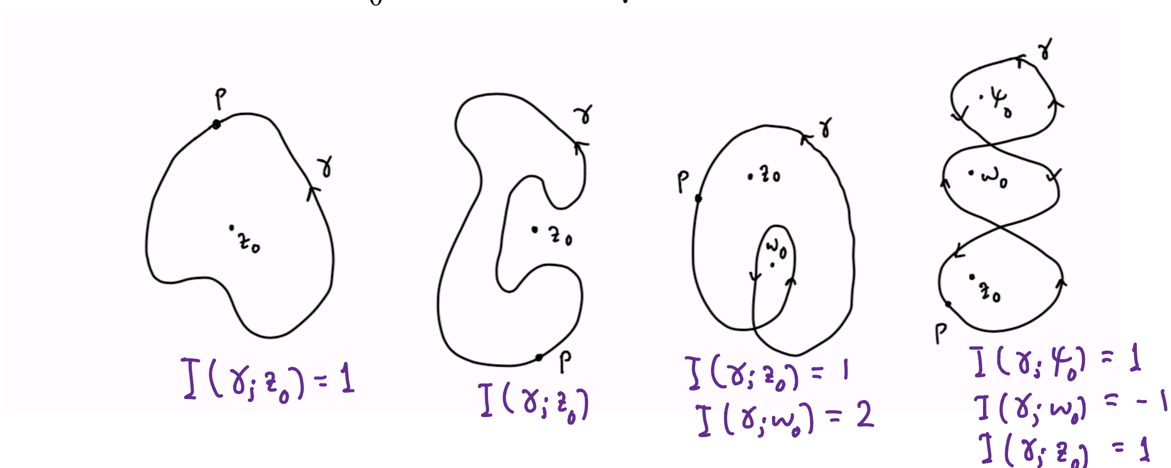
Math 4200
Monday October 5
2.4 Winding number (aka index), and Cauchy's integral formula

not on the exam. I'll try to post next HW set today or tomorrow

Announcements:
- talk more abt exam on Wed
- tentative plan:
  - sign an honor code on exam
  - closed book & closed everything!
  - you'll tell me a 2-hour window
    start btwn 10 - 4 on Fri:
    upload soltns to Gradescope w/th 2-hours

2.4 This section is about the magic fact that if a piecewise C^1 closed contour γ is given; and if $f(z)$ is analytic in an open simply-connected domain A containing γ, and if z_0 is "inside" γ, then $f(z_0)$ can be computed with an appropriate contour integral around γ. This is the *Cauchy Integral Formula* and is the basis for many amazing facts about analytic functions, and corollaries important in diverse pure and applied mathematics applications.

<u>Step 1</u> What does it mean for z_0 to be "inside" γ ?



Def If γ is a continuous closed path in $\mathbb{C}$, $\gamma : [a, b] \to \mathbb{C}$, $\gamma(a) = \gamma(b) = P$, and if $z_0 \notin \gamma([a, b])$ then the *winding number* of γ about z_0, also called the *index* of γ *relative to* z_0, is how many times γ winds around z_0 in the counterclockwise direction. We write $I(\gamma\,; z_0)$ for this integer.

- This number is usually easy to compute if you can see the image curve γ.

Examples: deduce the winding numbers of the various closed curves above, about the indicated points.

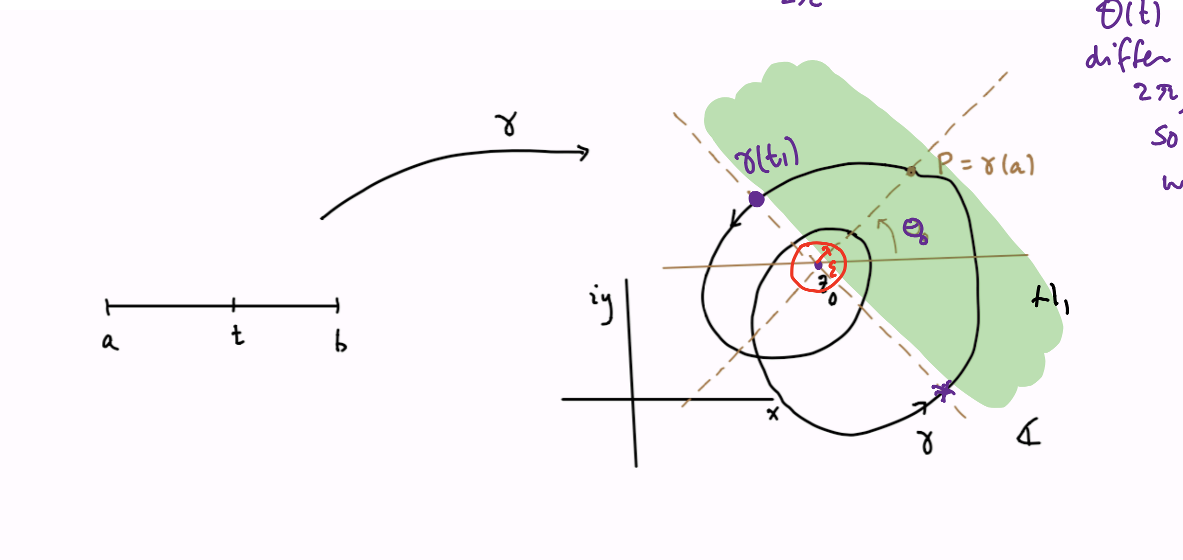
- *Def* We say that z_0 is *inside* γ iff $I(\gamma\,; z_0) \neq 0$.

Lemma $I(\gamma\,;z_0)$ is a well defined integer for any continuous closed curve γ with $z_0 \notin \gamma([a, b])$.

idea: we'll show that ~~of~~ we can define $\Theta(t)$ $a \le t \le b$ to be continuous choice of $\arg(\gamma(t) - z_0)$ then $I(\gamma; z_0) = \dfrac{\Theta(b) - \Theta(a)}{2\pi}$

proof:

(And all choices of $\Theta(t)$ fcn differ by mult of 2π, so $\Delta\Theta$ is well defined)



- $\gamma([a, b])$ is compact and contains its limit points, so $z_0 \notin \gamma([a, b])$ means γ stays a uniform distance away from z_0, $|z_0 - \gamma(t)| \ge \varepsilon > 0$.

- Claim (1): Make a choice θ_0 for the argument of $\gamma(a) - z_0$ (determined up to a multiple of $2\,\pi$). Then there is a unique way to extend $\theta = \theta(t) = arg(\gamma(t) - z_0)$ as a continuous function on the interval $[a, b]$, i.e. so that

$$\gamma(t) - z_0 = |\gamma(t) - z_0|\, e^{i\,\theta(t)} \qquad \forall\ t \in [a, b].$$

proof: Consider the open half plane half plane indicated above:

$$H_1 = \left\{ z \in \mathbb{C} \,\middle|\, \theta_0 - \frac{\pi}{2} < arg(z - z_0) < \theta_0 + \frac{\pi}{2} \right\}.$$

As long as $\gamma([a, t]) \subseteq H_1$ there is a unique way to define

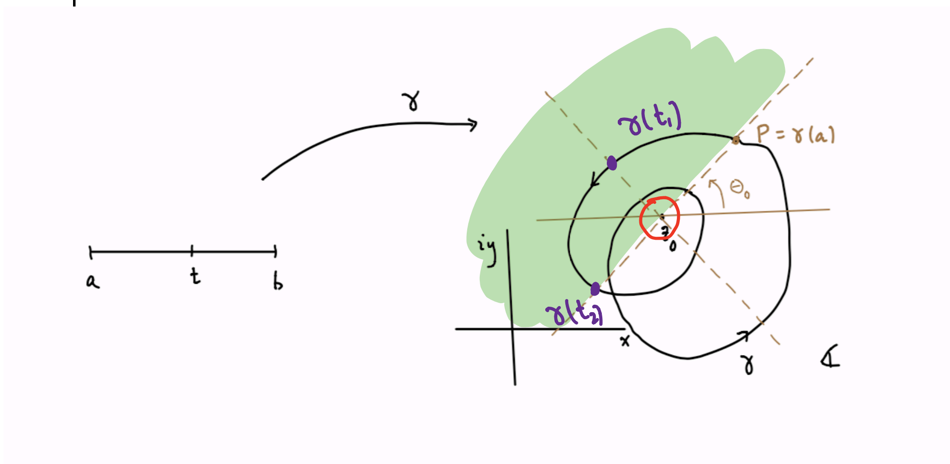
$$\theta(t) = arg(\gamma(t) - z_0)$$

continously, namely by requiring $\theta_0 - \dfrac{\pi}{2} < \theta(t) < \theta_0 + \dfrac{\pi}{2}\,\theta(t)$.) Let t_1 be the first $t > a$ with

$$\theta_0 - \frac{\pi}{2} = arg(\gamma(t_1) - z_0) \ \text{ or } \ \theta_0 + \frac{\pi}{2} = arg(\gamma(t_1) - z_0).$$

Then extend $\theta(t)$ for $t > t_1$ using the neighbor halfplane

$$H_2 = \left\{ z \in \mathbb{C} \;\middle|\; arg\big(\gamma(t_1) - z_0\big) - \frac{\pi}{2} < arg\big(z - z_0\big) < arg\big(\gamma(t_1) - z_0\big) + \frac{\pi}{2} \right\}.$$



Continue inductively, finding t_2, t_3 ,.... and half planes H_3, H_4 , ... if necessary. Because $\gamma(t)$ is uniformly continuous and because $|\gamma(t) - z_0| \geq \varepsilon$, this process terminates after a finite number of steps with

$$\theta(t) = arg\big(\gamma(t) - z_0\big)$$

$|\gamma(t_{k+1}) - \gamma(t_k)| > \varepsilon$
$\Rightarrow \exists\, \delta$ s.t. $|t_{k+1} - t_k| > \delta$
(unif cont)

defined and continuous on the entire interval $[a, b]$, and so that

$$\gamma(t) - z_0 = |\gamma(t) - z_0|\, e^{i\,\theta(t)}.$$

(2) Define $I(\gamma\,; z_0) := \dfrac{1}{2\pi}\big(\theta(b) - \theta(a)\big)$. Since $\gamma(a) = \gamma(b)$ and $\theta(b)$ and $\theta(a)$ are argument choices of $\gamma(a) - z_0$, the index is an integer. Any other continuous construction of the argument function, say $\theta_1(t)$ would have

$$\theta_1(t) - \theta(t) = 2\pi\, k(t), \quad k(t) \in \mathbb{Z}.$$

Since $k(t)$ is a difference of continuous functions it is continuous on $[a, b]$ and since it only takes on integer values, i.e. $\theta_1(t) = \theta(t) + 2\pi k$, it must be constant. Thus

$$\frac{1}{2\pi}\big(\theta_1(b) - \theta_1(a)\big) = \frac{1}{2\pi}\big(\theta(b) - \theta(a)\big)$$

so $I(\gamma\,; z_0)$ is well defined.

Theorem If γ, z_0 are as in the preceding discussion, and if γ is also piecewise C^1, then index can be computed with a contour integral:

$$\bullet \quad I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\, dz.$$

motivation: locally, in polar coordinates, $z = z_0 + r\,e^{i\theta}$

$$\bullet \quad z = z_0 + r\,e^{i\theta}$$

$$\bullet \quad dz = (dr)e^{i\theta} + r\,e^{i\theta} i\, d\theta$$

$$\Rightarrow \frac{dz}{z - z_0} = \frac{(dr)e^{i\theta} + r\,e^{i\theta} i\, d\theta}{r\,e^{i\theta}} = \frac{dr}{r} + i\,d\theta.$$

proof: Let $a \le s \le b$, θ_0, $\theta(t)$, H_1, t_1, H_2, t_2 ... as in the earlier discussion. If γ is C^1, then for $a \le s \le t_1$ and using polar coordinates in the intial half plane H_1,

$$\gamma(t) = z_0 + r(t)e^{i\theta(t)}$$

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\underbrace{\gamma(t)}_{z} - z_0}\, \underbrace{\gamma'(t)\, dt}_{dz} = \frac{1}{2\pi i}\int_a^s \frac{r'(t)e^{i\theta(t)} + r(t) i\, e^{i\theta(t)}\theta'(t)}{r(t)e^{i\theta(t)}}\, dt \qquad \text{det.}$$

$$= \frac{1}{2\pi i}\Big(\ln(r(s)) - \ln(r(a)) + i(\theta(s) - \theta(a))\Big).$$

Continue for $t_1 \le s \le t_2$, ... and adding more subintervals if γ is only piecewise C^1. Using telescoping series, deduce that for all $a \le s \le b$,

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\, \gamma'(t)\, dt = \frac{1}{2\pi i}\left(\ln\left(\frac{r(s)}{r(a)}\right) + i(\theta(s) - \theta(a))\right). \quad \bullet$$

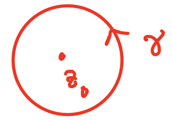
At $s = b$ for this closed curve $\ln\left(\frac{r(b)}{r(a)}\right) = 0$ so $\bullet$

$$\frac{1}{2\pi i}\int_a^s \frac{1}{\gamma(t) - z_0}\, \gamma'(t)\, dt = \frac{1}{2\pi}(\theta(b) - \theta(a)) = I(\gamma; z_0). \quad \bullet$$

Q.E.D.

Examples or $\gamma_{-n}(t) = z_0 + re^{-it}$

a) Show that for $\gamma_n(t) = z_0 + r\,e^{it}$, $0 \le t \le 2\,n\pi$, $n \in \mathbb{Z}$ that the winding number of n agrees the contour integral formula



$$I(\gamma\,;z_0) = \frac{1}{2\,\pi\,i}\int_{\gamma} \frac{1}{z - z_0}\,dz.$$

b) Let A be an open connected domain with boundary a p.w. C^1 simple closed curve γ oriented counterclockwise, $z_0 \in A$. Show

$$I(\gamma\,;z_0) = 1$$

via contour replacement.

a) $$\frac{1}{2\pi i}\int_{\gamma_n} \frac{1}{z-z_0}\,dz$$

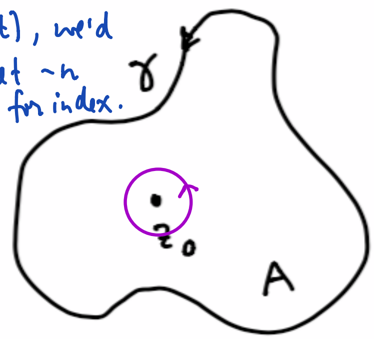
$$= \frac{1}{2\pi i}\int_0^{2\pi n} \frac{1}{\cancel{z_0}+re^{it}-\cancel{z_0}}\,rie^{it}\,dt$$

$$\gamma_n'(t) = rie^{it}\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi n} i\,dt$$

$$= \frac{1}{2\pi i}\,2\pi n i = n\,!!$$

If we used $\gamma_{-n}(t)$, we'd get $-n$ for index.



b) by §2.2 Deformation thm

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{1}{z-z_0}\,dz = \frac{1}{2\pi i}\oint_{|z-z_0|=\varepsilon}\frac{1}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\,2\pi i$$

$$= 1.$$

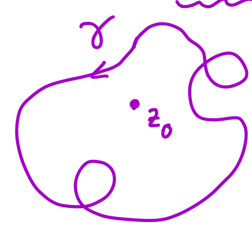
Remark If γ is any simple closed curve in $\mathbb{C} \setminus \{0\}$ then it turns out that γ is homotopic as closed curves to one of the γ_n'*s* in part (a). And so one could use the deformation theorem to deduce its index. This homotopy fact is typically proven in courses on algebraic topology where one studies the *fundamental group* of various spaces, including $\mathbb{C} \setminus \{0\}$ and the unit circle.

The Cauchy Integral Formula (which we will see is amazing in its consequences):

- Let $A \subseteq \mathbb{C}$ be open and connected , $f: A \to \mathbb{C}$ analytic;
- $\gamma: [a, b] \to \mathbb{C}$ a piecewise C^1 closed contour in A, $z_0 \notin \gamma([a, b])$.

Let γ be *contractible* in A, i.e. homotopic to a point as closed curves in A. (If A is simply connected this is automatic.)

Then



$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) I(\gamma\, ; z_0) .$$

!

(So, if z_0 is inside γ then $f(z_0)$ is determined and computable just from the values of f along γ !!!)

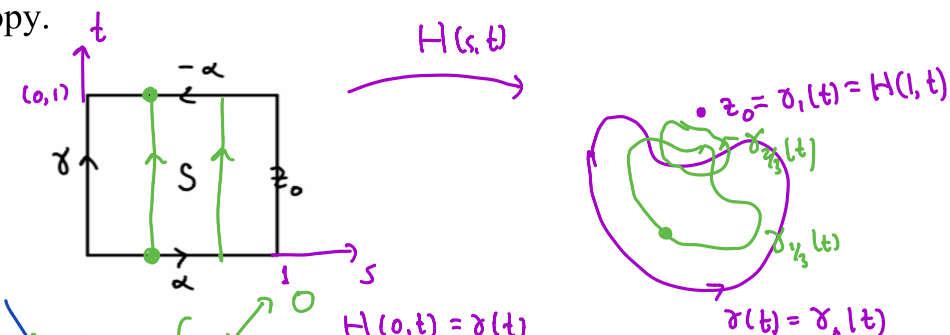
proof: Let

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

1) g is analytic in $A \setminus \{z_0\}$ and continuous at z_0.

2) So the modified rectangle lemma holds in any subdisk of A containing z_0 (see last Friday notes) and the local antiderivative theorem holds for $g(z)$.

3) So we may apply the *homotopy lemma* with the contractible curve diagram below, because it only depends on local antiderivatives and a subdivision argument in the domain and range of the homotopy.



to deduce $\int_\gamma g(z)\, dz = 0$. : $\int_\alpha \cancel{g(z)\,dz} + \int_{z_0} \cancel{g(z)\,dz} - \int_\alpha \cancel{g(z)\,dz} - \int_\gamma g(z)\,dz = 0$

4) Since z_0 is not on the countour defined by γ,

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0)\underbrace{2\pi i\, I(\gamma\, ; z_0)}$$

$$= \int_\gamma \frac{f(z)}{z - z_0}\, dz - \int_\gamma \frac{f(z_0)}{z - z_0}\, dz$$

Q.E.D.

Remark: If γ is a counter-clockwise simple closed curve bounding a subdomain B in A, with z_0 inside γ, then we already checked that $I(\gamma\,;z_0) = 1$. So in this case the CIF reads

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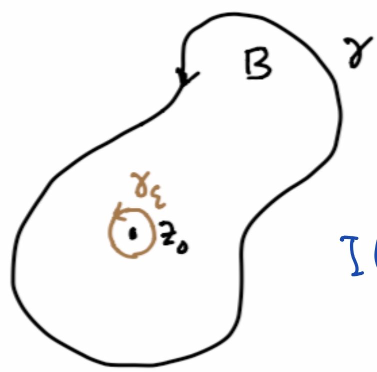
$$I(\gamma; z_0)\, f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

One can prove this important special case of the Cauchy integral formula with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic. This is a post-exam homework problem.

no quiz W.



$I(\gamma; z_0) = 1$

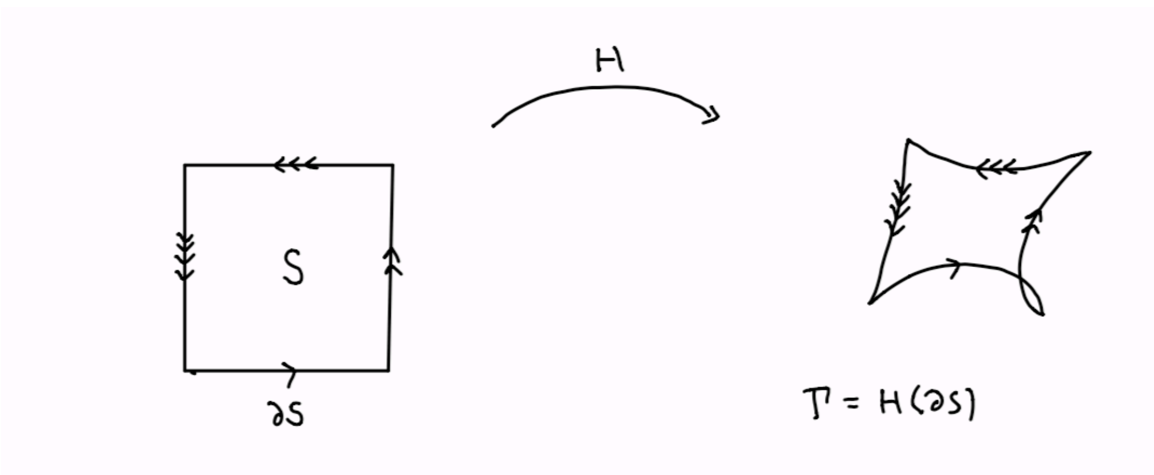
From Friday:

<u>Homotopy Lemma</u> Let $A \subseteq \mathbb{C}$ be open and connected. Let $f : A \to \mathbb{C}$ be analytic. Let

$$S = \{ (s, t) \mid 0 \le s \le 1,\ 0 \le t \le 1 \} \text{ and}$$
$$\delta S$$

denote the unit square and its boundary, oriented counterclockwise. Let $H : S \to A$ be continuous, with $\Gamma := H(\delta S)$ a piecewise C^1 contour. Then

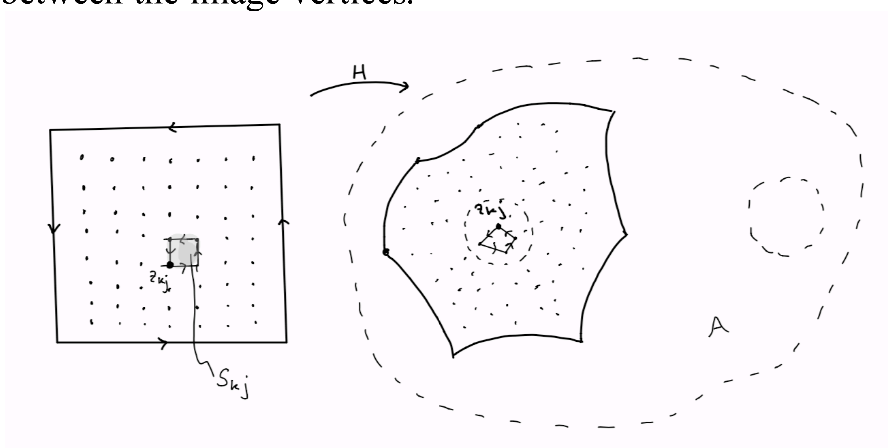
$$\int_\Gamma f(z)\, dz = 0.$$



proof of the homotopy lemma: Subdivide S into n^2 subsquares of side lengths n^{-1}. The dots in the diagram on the left indicate their vertices. number the squares as you would a matrix, and let $S_{k\,j}$ be a typical subsquare, with $\boldsymbol{z}_{k\,j}$ be the image under the homotopy of its lower left corner. Since H is continuous and S is compact, the image $H(S) \subseteq A$ is compact. Write

$$H(\delta\, S) = \Gamma$$
$$H(\delta\, S_{k\,j}) = \Gamma_{k\,j}.$$

Replace any of the four subarcs of each $\Gamma_{k\,j}$ which are not C^1 with constant speed line segment paths between the image vertices.



By interior cancellation,

$$\int_{\Gamma} f(\boldsymbol{z})\; d\,\boldsymbol{z} = \sum_{k,\,j} \int_{\Gamma_{k\,j}} f(\boldsymbol{z})\; d\,\boldsymbol{z}.$$

Note:

1) $H(S)$ is compact, $H(S) \subseteq A$ open, so by the Positive Distance Lemma you're proving in this week's homework

$$\exists\; \varepsilon > 0 \text{ such that } \forall\; \boldsymbol{z} \in H(S),\ \mathrm{D}(\boldsymbol{z}; \varepsilon) \subseteq A.$$

2) H is continuous on S so H is uniformly continuous. Thus for ε as in (1),

$$\exists\; \delta > 0 \text{ such that } \| (s, t) - (\tilde{s}, \tilde{t}) \| < \delta \Rightarrow |H(s, t) - H(\tilde{s}, \tilde{t})| < \varepsilon.$$

3) If n is large enough so that the diagonal length of the subsquares is less than δ, then each

$$H(S_{k\,j}) \subseteq \mathrm{D}(\boldsymbol{z}_{k\,j}; \varepsilon) \subseteq A,\ \boldsymbol{z}_{k\,j} = H(s_k, t_j).$$

4) By the local antidifferentiation theorem in $\mathrm{D}(\boldsymbol{z}_{k\,j}\,; \varepsilon)$, each

$$\int_{\Gamma_{k\,j}} f(\boldsymbol{z})\; d\,\boldsymbol{z} = 0 \Rightarrow \int_{\Gamma} f(\boldsymbol{z})\; d\,\boldsymbol{z} = 0. \qquad \text{Q.E.D.!!!}$$